Indian Musical Drum
Eigenspectra and Sound Synthesis

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CERTIFICATE

This is to certify that the project titled **Indian Musical Drum - Eigenspectra and Sound Synthesis**, submitted by **Ganesh Saraswat (ME06B107)**, to the Indian Institute of Technology, Madras, for the award of the degrees of **Bachelor of Technology in Mechanical Engineering and Master of Technology in Product Design**, is a bonafide record of the project work done by him in the Department of Mechanical Engineering, IIT Madras. The contents of this report, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

KEYWORDS: Tabla; Physical modelling; Sound synthesis; Damping; Pseudospectra; Modal synthesis.

In the field of digital sound synthesis people have long endeavored to digitally recreate the sounds of traditional musical instruments. Physical modeling based synthesis has come to the fore with increased computing power. It involves the physical description of the musical instrument in terms of a set of coupled partial differential equations. An output waveform is obtained through numerical resolution of such a system subject to some form of excitation. The advantage is that the model parameters are physically meaningful, intuitive and relate to the instruments geometry, material properties and the forces acting on it.

Indian musical drums with *Tabla* as prototypical example have a circular drum head that is of non-uniform density. Prior investigations have remarkably revealed that the low eigenmodes are harmonic. In this work a bi-density membrane model with stepped variation in density is considered first. A Green's function based technique is then used to estimate the effects of air loading on such a bi-density membrane. Next the model for a smoothly varying non-uniform membrane is presented which accounts for eccentric loading; phenomenological damping terms are used to model the loss characteristics. A Fourier-Chebyshev spectral collocation method is used to obtain highly accurate eigenvalues and eigenfunctions. Finally an attempt is made at numerical sound synthesis for *Tabla* using Modal Synthesis.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS **i**

ABSTRACT **ii**

LIST OF FIGURES **v**

ABBREVIATIONS **vi**

NOTATION **vii**

1 INTRODUCTION **1**

1.1 Literature Review 1

1.2 Objective 3

1.3 Organization of thesis 4

2 AN ANALYTICAL APPROACH **5**

2.1 The Tabla Drumhead 5

2.2 The bi-density circular membrane 5

2.3 Radiation of sound and associated damping 8

2.3.1 Green's function approach for radiation damping 9

2.3.2 Determining eigenmodes for damped bi-density membrane . 10

2.3.3 Cubature for Numerical resolution of the eigensystem . . . 13

2.4 Results and Discussion 14

3 A NUMERICAL APPROACH **16**

3.1 Membrane with smoothly varying density 16

3.2 Lossy wave equation for non-uniform membrane 17

3.3 Computing the Eigenspectra 18

3.4 Results 21

4 NUMERICAL SOUND SYNTHESIS **22**
4.1 Modal Synthesis 22
4.2 Mathematical formulation 23
4.2.1 Raised cosine initial condition 24
4.2.2 Propagation in time 25
4.3 Results and Discussion 25

5 SUMMARY AND CONCLUSION **29**

6 REFERENCES **30**

LIST OF FIGURES

2.1 The drumhead of the *bayan* and the *dayan* (right). 5

2.2 Visualisation of the eigenvalues as points of intersection. With parameters $\sigma = 3.125,\ k = 0.4$ successive eigenvalues are obtained for $m = 0.$ 7

2.3 Comparison of the eigenfrequencies for the damped and undamped cases of homogeneous and bi-density membranes 14

2.4 Eigenmodes for the damped bi-density case 15

2.5 Variation of the eigenfrequencies with mass density ratio σ for the concentric case 15

3.1 The smoothly varying density model for *dayan* and *bayan* 17

3.2 The variation of areal density with the radial coordinate r for *dayan* and *bayan* respectively 17

3.3 The Fourier-Chebyshev Grid 19

3.4 Damped Eigenmodes for the concentric case 21

3.5 Damped Eigenmodes for the eccentric case 21

4.1 The raised cosine initial condition. 24

4.2 Output waveform and Power Spectral Density of Tabla bol *'Tun'* . . 26

4.3 Spectrogram output for the Tabla bol *'Tun'* 27

4.4 Propagation of 2D wave on membrane 27

4.5 Numerically synthesized sound sample 28

ABBREVIATIONS

IIT	Indian Institute of Technology
PDE	Partial Differential Equation
PSD	Power Spectral Density

NOTATION

r	Radius, m
ϕ	Azimuthal angle, rad
a	Inner radius of bi-density membrane, m
b	Outer radius of bi-density membrane, m
T	Tension, N
ρ	Density, $kg\, m^{-3}$
c	Wave speed on membrane, $m\, s^{-1}$
ω	Angular frequency, $rad\, s^{-1}$
κ	Wave number, m^{-1}
m	Order of the Bessel function; nodal lines
n	Order of eigenvalue; nodal circles
k	Radius ratio
σ	Density ratio
Ψ_{mn}	Eigenfunction for the membrane
c_a	Speed of sound in air, $m\, s^{-1}$
η	Displacement of membrane, m
p	Pressure, Pa
$\eta_{ms}(\rho, \phi)$	Eigenmode for the of the loaded bi-density membrane, m
$\eta_{mn}^{(0)}(\rho, \phi)$	Eigenmode for the in vacuo bi-density membrane, m
W	Weighing function for orthogonality relation
ϵ	Eccentricity parameter
ξ	Density variation parameter
γ	Spatially scaled wave speed , s^{-1}
u	Displacement of the smoothly non-uniform membrane, m
σ_0 σ_1	Damping parameters, $s^{-1}, m^2 s^{-1}$
$\|\mathbf{\Psi_{in}}\rangle$	*Ket* vector representing initial shape of membrane
$\|\mathbf{\Psi_i}\rangle$	*Ket* vector representing an eigenmode
$\|\mathbf{\Psi(t)}\rangle$	*Ket* vector representing displacement at time t

CHAPTER 1

INTRODUCTION

Eigenvalue problems for a string and a uniform circular membrane are classical problems in mathematical physics. The eigenvalues of a string are determined by the zeros of the sine function and so form a harmonic series. The large number of harmonic overtones gives the vibrations of a string its musicality. The eigenvalues of a uniform membrane, on the other hand, are determined by the zeros of Bessel functions. The overtones are not integer multiples of the fundamental. Consequently, the vibrations do not have a strong sense of pitch and, therefore, lack the musicality of string vibrations.

Several musical traditions have devised means of restoring musicality to the vibrations of circular drums. The Western Timpani achieve this by coupling the vibrations of the membrane with the large mass of air enclosed in the kettle below the drum head. For a judicious choice of modes, the combined membrane-air system has harmonic vibrations [1]. A different strategy is used in a whole family of drums used in the Indian subcontinent (notably *Mirdangam* and *Tabla*), where harmonic overtones are obtained by loading the central part of the membrane with material of heavier density.

Earlier theoretical attempts were made to model the vibration of such drums and explain their remarkable tonal quality. In this thesis a physical model for the *Tabla* drumhead is developed and used in synthesizing sound numerically.

1.1 Literature Review

In this section extensive work carried out by researchers on the family of Indian drums is discussed. Furthermore in connection with the aim of the thesis a discussion of works investigating the effects of air loading and other mechanisms of loss for a vibrating membrane is also presented.

Raman made the first scientific study of the family of Indian drums [2]. In a series of experiments, Raman and coworkers obtained the eigenmodes and eigenvalues of the Mridangam, showing that the first nine normal modes gave five very nearly harmonic tones. The higher overtones were noticeably inharmonic, but Raman noted features in the construction of design to suppress the higher overtones.

Subsequently, Ramakrishna and Sondhi modelled the drum head as a composite membrane of two distinct densities [3]. With this simplification, and for concentric loading, the eigenvalue problem could be solved analytically in terms of Bessel and trigonometric functions.

A solution for the composite membrane model with eccentric loading due to lack of circular symmetry is considerably more difficult. An exact solution for the eigenmodes in terms of known functions is not available. Two approximate solutions one by Ramakrishna [4] and other due to Sarojini and Rahman [5] were presented but agreed poorly with experimental values.

Attempts were made to model the drumhead by continuous densities; notably by Ghosh [6] and Rao [7]. Their work focussed only on the concentric case and used rather unphysical models for the mass densities.

Sathej and Adhikari [8] recently developed a model where the density of the loaded region varies gradually. They used a continuous model of density variation and by applying a Spectral Collocation method determined the eigenvalues and eigenmodes to a high degree of accuracy.

The methods above aimed at characterizing the eigenspectra leave out the effect of radiation damping which is important for a more realistic physical model. Christian et al applied the Green function method for the case of Timpani membrane to calculate the effect of air loading on its vibrations and computed its modal frequencies and decay times [9]. They obtained corrections for the modal frequencies that were in good agreement with values determined through their experiments but the decay times compared poorly.

At low frequencies however other sources of loss such as thermoelasticity and viscoelasticity become important. A detailed picture of such effects for damped impacted plates is presented by Antoine and Lambourg [10].

In his authoritative book on Numerical Sound Synthesis Bilbao discusses a phenomenological model with explicit damping terms which appears more suitable from sound synthesis perspective [11]. Bilbao provides implementations of Finite-Difference schemes for applications to such models.

Trefethen in his book on Spectral Methods [12] discusses their application in solving varying forms of PDEs. He provides concise illustrative MATLAB codes to solve problems some of which are typical eigenvalue problems.

An opportunity is seen to investigate the eigenspectra of Tabla while considering damping effects. Furthermore with the aid of Spectral Methods an accurate physical model can be realised leading to numerical sound synthesis.

1.2 Objective

The major objective of the work undertaken for this thesis is to develop a physical model for the *Tabla* drumhead leading to Numerical Sound Synthesis. As part of the this study damping effects are considered to obtain a more realistic model of vibration.

Using an analytical approach wherein the *Tabla* drumhead is modeled as a bi-density membrane with stepped variation in density the the effect of radiation damping is investigated. Next the variation of harmonicity with density and area ratios is to be characterized.

As a refinement a numerical approach in which the drumhead is modeled as a smoothly non-uniform membrane will be considered which would account for both concentric and the eccentric case. A phenomenological damping model that accounts for the energy loss will then be applied.

With the eigenvalues and eigenfunctions determined for the model a modal synthesis technique will be used to generate an output waveform subject to initial excitation.

1.3 Organization of thesis

This thesis is divided into five chapters. Chapter 1 begins with a review of the literature survey undertaken for this thesis. Next the scope and aim of this thesis were established.

Chapter 2 begins with an introduction to *Tabla* discussing various features with regard to its construction, musical quality and remarkable tonal characteristics. This paves the way for the presentation of a bi-density model of membrane. The bi-density model is described in detail and mathematical formulation of its vibration is presented. This is followed by the development of the Green's function analytical technique that accounts for radiation damping in a membrane-air coupled system. The numerical resolution for the damped bi-density membrane is covered and the chapter concludes with results and their discussion.

Chapter 3 explores a purely numerical approach to the modelling of Tabla drumhead as a lossy membrane. A smoothly varying density model for the membrane is then presented which is followed by introduction of PDE containing explicit phenomenological damping terms that describe the decaying vibrations for such a membrane. A mathematical approach based on Spectral methods is discussed next for a numerical solution of the PDE. A matrix formulation for the *Quadratic Generalized Eigenvalue* problem is developed concluding with results and discussion.

Chapter 4 presents a numerical technique based *Modal Synthesis* for realizing sound from the model developed for a lossy smoothly non-uniform membrane. The mathematical formulation for modal synthesis based on a physical model is developed covering initial excitation and propagation of 2D wave on the membrane in time. The chapter concludes with results and discussion of the simulations.

Chapter 5 concludes the thesis by giving a summary of work done and specifying the scope of future work.

CHAPTER 2

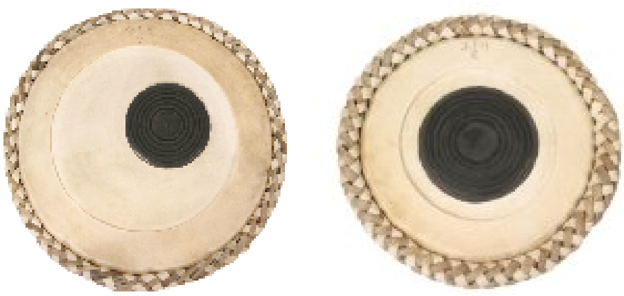
AN ANALYTICAL APPROACH

2.1 The Tabla Drumhead

The Tabla drumhead is actually a composite membrane which broadly consists of three components, the *maidan*, the *chat* and the denser, black central region called the *syahi*. The *syahi* is made of a paste of soot, iron filings and flour and applied layer by layer to the goatskin membrane. The *syahi* is later cracked with a stone after it has dried to reduce its rigidity.The *chat* is an upper annular layer of skin which covers only the outer periphery of the opening. The *maidan* is the only layer that covers the entire opening. The *syahi* is the most distinctive part of *Tabla* and gives it its distinctive tone.

2.2 The bi-density circular membrane

The previous section described the complex structure of the Tabla drumhead. The distinctive tonal quality of the Tabla can be attributed to its unique construction. Previously, various models have been proposed that idealize the drumhead as a membrane with either varying thickness or density. We begin with the description of the bi-density



(a) (b)

Figure 2.1: The drumhead of the *bayan* and the *dayan* (right).

model by Ramakrishna and Sondhi that serves as the starting point for our investigations.

The drumhead is modelled as a circular membrane of radius $r = b$ clamped at the boundary with areal densities $\rho_1(0 \le r < a)$ and $\rho_2(a \le r \le b)$ and uniform tension T per unit length. The transverse displacement ψ of the membrane satisfies wave equation over the circular domain

$$\nabla^2\psi = \begin{cases} 1/c_1^2 \dfrac{\partial^2\psi}{\partial t^2} & 0 \le r < a \\ 1/c_2^2 \dfrac{\partial^2\psi}{\partial t^2} & a \le r \le b \end{cases} \quad \text{where} \quad c_1^2 = T/\rho 1\ , \quad c_2^2 = T/\rho 2$$

Looking for stationary solutions of the form

$$\psi = \Psi\, e^{\imath\omega t} = \begin{cases} \Psi_1(r,\phi)\, e^{\imath\omega t} & 0 \le r < a \\ \Psi_2(r,\phi)\, e^{\imath\omega t} & a \le r \le b \end{cases}$$

and substituting them into the wave equation we obtain the Helmholtz equations

$$\nabla^2\Psi_1 + \kappa_1^2\,\Psi_1 = 0 \quad 0 \le r < a$$
$$\nabla^2\Psi_2 + \kappa_2^2\,\Psi_2 = 0 \quad 0 \le r < a$$

where $\quad \kappa_1 = \omega/c_1\ , \quad \kappa_2 = \omega/c_2$

The solutions Ψ_1 and Ψ_2 and their derivatives need to be continuous at $r = a$. Boundary conditions require periodicity in the angular variable ϕ and Ψ_1 and Ψ_2 to be zero at $r = b$. These conditions can be expressed as

$$\Psi_1(a,\phi) = \Psi_2(a,\phi)$$
$$\frac{\partial\Psi_1}{\partial r}(a,\phi) = \frac{\partial\Psi_1}{\partial r}(a,\phi)$$
$$\Psi_2(b,\phi) = 0$$
$$\Psi_i(r,\phi) = \Psi_i(r, 2n\pi + \phi) \qquad i = 1,2$$

The general solution, found in numerous texts (for example by Morse Ingard) is given in terms of Bessel functions and we write

$$\Psi_1(r,\phi) = A_m J_m(\kappa_1 r) e^{\imath m\phi}$$
$$\Psi_2(r,\phi) = [B_m J_m(\kappa_2 r) + C_m Y_m(\kappa_2 r)] e^{\imath m\phi}$$

Y_n term in Ψ_1 is dropped for reasons of finiteness. On application of boundary conditions one obtains, for different vales of m, the following eigenvalue problem

$$\sigma\frac{J_{m-1}(\sigma kx)}{J_{m-1}(\sigma kx)} = \frac{J_{m-1}(kx)Y_m(x) - J_m(x)Y_{m-1}(kx)}{J_m(kx)Y_m(x) - J_m(x)Y_m(kx)}$$

with dimensionless quantities

$$x = \kappa_2 b = \omega b/c_2\,; \quad \text{density ratio}\,\sigma^2 = \frac{\rho_1}{\rho_2} = \frac{\kappa_1^2}{\kappa_2^2} = \frac{c_2^2}{c_1^2}\,; \quad \text{radius ratio}\,k = \frac{a}{b}$$

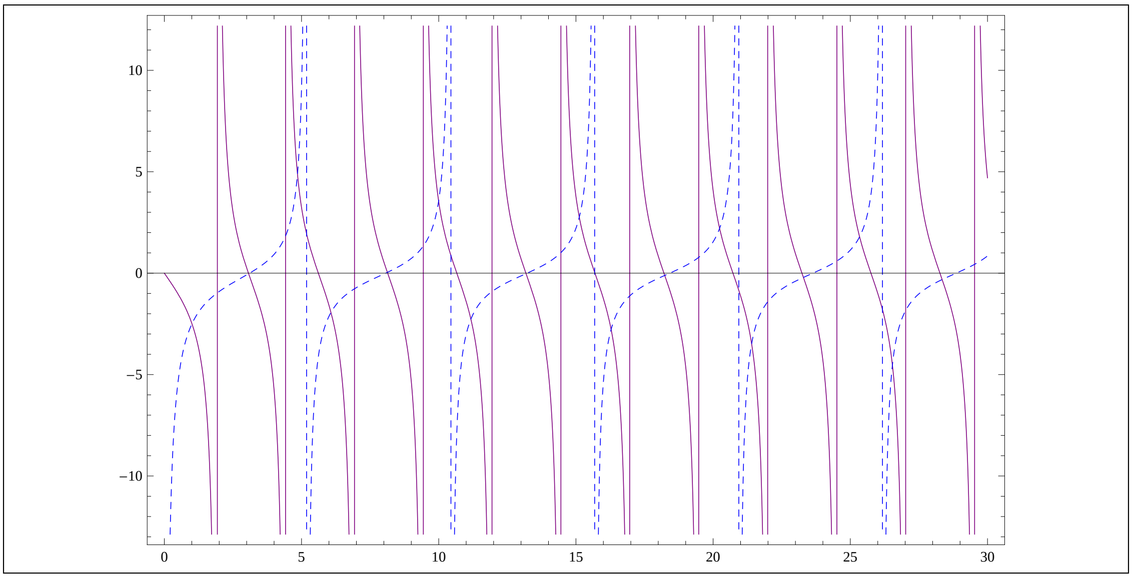


Figure 2.2: Visualisation of the eigenvalues as points of intersection. With parameters $\sigma = 3.125$, $k = 0.4$ successive eigenvalues are obtained for $m = 0$.

With x_{mn} denoting the n^{th} eigenvalue for the m^{th} mode the coefficients are determined to be

$$B_{mn} = A_{mn}\frac{J_m(\sigma k x_{mn})Y_m(x_{mn})}{J_m(kx_{mn})Y_m(x_{mn}) - J_m(x_{mn})Y_m(kx_{mn})}$$
$$C_{mn} = -A_{mn}\frac{J_m(\sigma k x_{mn})J_m(x_{mn})}{J_m(kx_{mn})Y_m(x_{mn}) - J_m(x_{mn})Y_m(kx_{mn})}$$

The eigenfunctions can now be written in terms of arbitrary A_{mn}'s as

$$\Psi_{mn} = \begin{cases} \Psi_{1mn}(r,\phi) = A_{mn} J_m(\sigma k x_{mn} r/a) e^{\imath m\phi} \\ \Psi_{2mn}(r,\phi) = A_{mn} \dfrac{J_m(\sigma k x_{mn})}{J_m(k x_{mn}) Y_m(x_{mn}) - J_m(x_{mn}) Y_m(k x_{mn})} \\ \qquad\qquad \times\, J_m(k x_{mn} r/b) Y_m(x_{mn}) - J_m(x_{mn}) Y_m(k x_{mn} r/b) e^{\imath m\phi} \end{cases}$$

The eigenfunctions Ψ_{mn} are orthogonal and satisfy the orthogonality relation

$$\int_0^a \int_0^{2\pi} \sigma^2\, \Psi_{1mn} \Psi_{1m'n'}\, r\, \mathrm{d}r\mathrm{d}\phi + \int_a^b \int_0^{2\pi} \Psi_{2mn} \Psi_{2m'n'}\, r\, \mathrm{d}r\mathrm{d}\phi = 0; \quad m \neq m^{'}, \quad n \neq n^{'}$$

2.3 Radiation of sound and associated damping

When one talks about sound from a percussion instrument like Tabla or in our case its approximation as a clamped circular bi-density membrane it implies the existence of a fluid medium which is air that surrounds it. The motion of the membrane is not independent of the medium that surrounds it but is affected by certain kinematical and dynamical constraints. This is to say that the membrane and the surrounding air constitute a coupled system. This interaction is described in two parts as :

1. The vibrating membrane induces motion in the surrounding air generating pressure fluctuations which we perceive as sound.
2. The fluid in motion acts back on the membrane producing fluctuating forces at the interface. The motion of the membrane is modified as a result.

The sound radiated by the vibrating membrane continuously carries energy away from it in the form of sound waves giving rise to the what is termed as radiation damping. Other mechanisms of energy loss like *viscoelasticity* and *thermoelasticity* do exist and are also important but are left out of consideration for now. The analytic modelling of the coupled fluid-membrane system would proceed by writing down of equations of motion for the system constituents supplemented with coupling conditions at the interface. The vibrations of the membrane are assumed to be small with the surrounding air always in contact with it.

Theoretical attempts have been made earlier in the context of radiation damping for the Western *Timpani* or the *Kettledrum*. The focus was primarily on how enclosed-air in a cavity below the membrane affects its modal frequencies. Morse determined the frequency shifts for the axially symmetric modes Ψ_{0n} through a simplifying assumption of uniform pressure loading on the membrane. Benade looked at the modes with one nodal line (Ψ_{m1}) termed the *sloshing modes* which underscored their importance in bringing frequencies closer to a harmonic relationship.

In the context of the Tabla the cavity is of much smaller size enclosing a small volume of air. The role of such a small the cavity, though, not to be discounted seems unlikely to account for the remarkable tonal quality that the Tabla is known for. A major contribution to this end is therefore expected from the centrally loaded membrane itself. As a simplification one can consider outside air loading on a circular membrane with a stepped variation in density. A Green's function technique adapted from [9] is presented to model the air loading.

2.3.1 Green's function approach for radiation damping

The membrane of radius $r = b$ is surrounded by air on both sides. The incremental pressure p satisfies the wave equation,

$$\left(\nabla^2 - \frac{1}{c_a^2}\frac{\partial^2}{\partial t^2}\right) p = 0 \qquad c_a \text{ is the speed of sound in air}$$

looking for stationary solutions of the form $p = p(\mathbf{r})\, e^{\imath\omega t}$ gives us the Helmholtz equation.

$$\left(\nabla^2 + \frac{\omega^2}{c_a^2}\right) p(\mathbf{r}) = 0$$

We consider a Green's function $G(\mathbf{r}|\mathbf{r}')$ which satisfies the equations

$$\left(\nabla^2 + \frac{\omega^2}{c_a^2}\right) G(\mathbf{r}|\mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \quad \text{or}$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c_a^2}\right] G(\rho,\phi,z|\rho',\phi',z') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\delta(z-z')$$

An appropriate form of the Green's function $G(\mathbf{r}|\mathbf{r}')$ for the Helmholtz equation is given by

$$G(\mathbf{r}|\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\, e^{\imath\frac{\omega}{c_a}|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi|\mathbf{r}-\mathbf{r}*'|}\, e^{\imath\frac{\omega}{c_a}|\mathbf{r}-\mathbf{r}*'|}$$

$$|\mathbf{r}-\mathbf{r}'| = [\rho^2+\rho'^2-2\rho\rho' cos(\phi-\phi')+(z-z')^2]^{\frac{1}{2}}$$

$$|\mathbf{r}-\mathbf{r}*'| = [\rho^2+\rho'^2-2\rho\rho' cos(\phi-\phi')+(z+z')^2]^{\frac{1}{2}}$$

It can be derived using the method of images. The Green's function has vanishing normal derivative in the membrane plane i.e. $(\nabla G(\mathbf{r}|\mathbf{r}')|_{z'=0})\,.\,e_z = 0$

$$\nabla G(\mathbf{r}|\mathbf{r}') = 2\,\frac{e^{\imath\frac{\omega}{c_a}|r-r'|}}{4\pi|r-r'|} \quad \text{at} \quad z'=0$$

Using the second form of Green's theorem and the boundary conditions the pressure over the infinite region $p(x,y,z<0)$ can be written as

$$p(x,y,z<0) = \frac{1}{2\pi}\frac{\partial}{\partial z}\int \mathrm{d}x' \int \mathrm{d}y'\, \frac{e^{\imath(\frac{\omega}{c_a})\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\; p(x',y',0_-) \text{ also}$$

$$\frac{\partial p}{\partial z}(x,y,z<0) = \frac{1}{2\pi}\frac{\partial^2}{\partial z^2}\int \mathrm{d}x' \int \mathrm{d}y'\, \frac{e^{\imath(\frac{\omega}{c_a})\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\; p(x',y',0_-)$$

where $p(x',y',0_-)$ is the pressure on the bottom surface of the membrane and we have $p(x',y',0_-) = -p(x',y',0_+)$

2.3.2 Determining eigenmodes for damped bi-density membrane

The transverse displacement of the membrane η and pressure p are assumed to have harmonic time dependence

$$\eta = \eta(\rho,\phi)\, e^{-\imath\omega t} \;\; ; \qquad p = p(\rho,\phi)\, e^{-\imath\omega t}$$

The displacement η satisfies the wave equation with added pressure loading terms

$$\sigma(\rho)\,\frac{\partial^2\eta}{\partial t^2} = -\sigma(\rho)\,\omega^2\eta = T\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]\eta + p(\rho,\phi,0_-,t) - p(\rho,\phi,0_+,t)$$

σ is the areal mass density and T is the membrane tension per unit length.

The coupling condition at the interface is derived from linearized fluid dynamics

$$\rho_0 \, \ddot{\eta} = -\frac{\partial p}{\partial z}(\rho, \phi, z = 0, t) \qquad \rho_0 \text{ is the equilibrium density of air}$$

The membrane displacement is now related to the normal pressure gradient on its surface

$$\eta(x, y) = \frac{1}{\rho\omega^2}\frac{\partial p}{\partial z}(x, y, 0) = \frac{-1}{2\pi\rho_0\omega^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c_a^2}\right)\int \mathrm{d}x' \int \mathrm{d}y'$$
$$\times \frac{e^{\imath(\frac{\omega}{c_a})\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\; p(x', y', 0_-)$$

$$p(x', y', 0_-) = \frac{1}{2}\left[-\sigma^2(\rho)\omega^2\eta(x', y') - T\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\eta(x', y')\right]$$

On substitution an integro-differential equation is obtained which can be compactly written as,

$$\eta(\rho, \phi, \omega) = \frac{1}{\rho_0\omega^2}\left(\nabla^2_{\rho,\phi} + \frac{\omega^2}{c_a^2}\right)\int_0^b \rho \, \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \; G_{free}(\rho, \phi, z|\rho'\phi'z')$$
$$\times \left(\sigma(\rho)\,\omega^2 - T\,\nabla^2_{\rho'\phi'}\right)\eta(\rho', \phi', \omega)$$

Without the pressure loading terms the equation for the membrane displacement is representative of the *in vacuo* bi-density case. We assume the same azimuthal dependence $e^{\imath m\phi}$ for the loaded-membrane displacement $\eta_{ms}(\rho, \phi)$ and expand it in eigenmodes $\eta^{(0)}_{mn}(\rho, \phi)$ determined for the bi-density case.

$$\eta_{ms}(\rho, \phi) = \sum_{n=1}^{\infty} a^{(m)}_{n,s} \; \eta^{(0)}_{mn}(\rho, \phi) \quad m = 0, 1, 2, ... \quad s = 1, 2, 3, ...$$

The *in vacuo* bi-density eigenmodes $\eta^{(0)}_{mn}(\rho, \phi)$ satisfy an orthogonality relation. Substituting the above expansion into the integro-differential equation, multiplying with $W(\rho)\,\eta^{(0)*}_{mn}(\rho, \phi)$ and integrating over the membrane gives an infinite set of equations for $a_{n,s}^{(m)}$. These can be written down as,

$$a_{n,s}^{(m)} = \sum_{n=1}^{\infty} a_{n',s'}^{(m)} \frac{T}{4\pi\rho_0\omega^2}(\omega^2 - \omega_{mn}^2) \\ \times \left\{ \left(\frac{\omega^2}{c_a^2} - \frac{\omega_{mn}^2}{c_{m1}^2} \right) I_{m,n,n',\omega} + \left(\frac{\omega^2}{c_a^2} - \frac{\omega_{mn}^2}{c_{m2}^2} \right) \bar{I}_{m,n,n',\omega} - K_{m,n,n',\omega} \right\}$$

where $\quad c_{mi} = \sqrt{\frac{T}{\rho_i}} \quad i = 1, 2$

$$I_{m,n,n',\omega} = \int_0^{2\pi} \mathrm{d}\phi' \int_0^b \rho' \mathrm{d}\rho' \int_0^{2\pi} \mathrm{d}\phi \int_0^b \rho \, \mathrm{d}\rho \;\; \frac{4\pi\, W(\rho)}{c_m^2(\rho')} \;\; \eta_{mn}^{(0)*}(\rho, \phi) \\ \times G_{free}(\rho, \phi, z|\rho'\phi'z') \; \eta_{mn'}^{(0)}(\rho', \phi')$$

$$\bar{I}_{m,n,n',\omega} = \int_0^{2\pi} \mathrm{d}\phi' \int_0^b \rho' \mathrm{d}\rho' \int_0^{2\pi} \mathrm{d}\phi \int_0^b \rho \, \mathrm{d}\rho \;\; \frac{4\pi\, W(\rho)}{c_m^2(\rho')} \;\; \eta_{mn}^{(0)*}(\rho, \phi) \\ \times G_{free}(\rho, \phi, z|\rho'\phi'z') \; \eta_{mn'}^{(0)}(\rho', \phi')$$

$$K_{m,n,n',\omega'} = \int_0^{2\pi} \mathrm{d}\phi \int_0^b \rho' \mathrm{d}\rho' \;\; 8\pi^2 b \, \frac{W(\rho')}{c_m^2(\rho')} \, \frac{\partial}{\partial\rho} \eta_{mn}^{(0)*}(\rho, \phi) \bigg|_{\rho=b} \\ \times G_{free}(\rho, \phi, z|\rho'\phi'z') \; \eta_{mn'}^{(0)}(\rho', \phi')$$

The eigensystem is nonlinear in ω with coefficients that are integrals involving it. The approximate determination of low frequency modes may be accomplished by,

1. Limiting n, n' to the values 1,2,3 and 4
2. Substituting ω_{mn} for ω in $I_{m,n,n',\omega}$, $I_{m,n,n',\omega}$ and $K_{m,n,n',\omega'}$
3. Solving the truncated equations as a linear eigenvector-eigenvalue problem
4. Picking out the eigenfrequency ω for which the associated η_{ms} has n nodal circles and substituting it back in $I_{m,n,n',\omega}$, $I_{m,n,n',\omega}$ and $K_{m,n,n',\omega'}$
5. Repeating the steps until successive eigenfrequencies agree to 0.5%

2.3.3 Cubature for Numerical resolution of the eigensystem

The eigenvector-eigenvalue problem obtained for the damped bi-density membrane is solved iteratively as outlined in the previous section. The linear equations constituting the truncated eigensystem contain terms with four dimensional integrals over the circular domain. The existence of free space Green's function in the integrals makes them weakly singular. The difficulty in evaluation of these multidimensional integrals is compounded by their weakly singular nature.

Several techniques exist for evaluation of such weakly singular integrals. The method of singularity subtraction proceeds by transforming the integrand by subtracting the singular part and integrating it numerically. The subtracted singular part is then evaluated analytically. Another approach is to transform the integrand through an appropriate change of variable so that the singularity now occurs only at the end points of the domain. There are methods that discretize the domain into elements and use specialized schemes to evaluate the singular integral over them.

The simplest method however is to modify the singular $\frac{1}{r}$ term by adding a small constant ϵ. The new term $\frac{1}{r+\epsilon}$ is not singular as $r \to 0$. This is sufficient in our case to ameliorate the issues of singularity that arise when using a general purpose numerical integrator.

Use of quadrature to evaluate the multidimensional integrals through nesting is computationally highly inefficient. To obtain a quick evaluation with reasonable accuracy Monte-Carlo based cubature is used. We use the freely available cubature library *CUBA* for our implementation in *C*. To iterate over the σ, k parameter space significant computational time is needed so the need for parallelization arises. The code was parallelized with OpenMPI and use was made of the High Performance Computing Environment (HPCE) at IIT Madras. The remaining calculations were done using the symbolic computation package *MATHEMATICA*

2.4 Results and Discussion

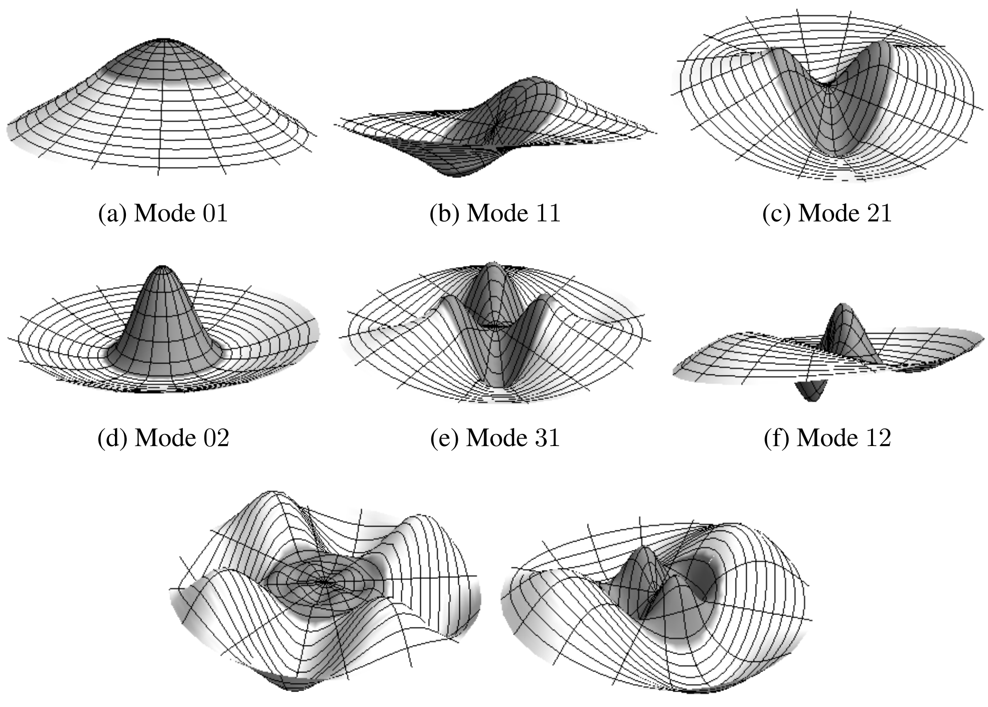
The figure below details the effect of radiation damping on homogeneous and bi-density membranes. For a homogeneous membrane, radiation damping results in significant lowering of the frequencies. Remarkably, for a bi-density membrane the effect is not pronounced and it results in only marginal reduction. It is observed that for a bi-density membrane the density variation, by itself, lowers the frequencies to levels comparable to a damped homogeneous case. One can see that the lower nine eigenmodes are nearly harmonic for the damped bi-density case.

When one iterates over the σ parameter space it is seen that for density ratios between 2 and 3 the harmonicity is lifted to a great extent due to radiation damping effects. However, as the density ratio becomes higher the overtones exhibit convergence but remain inharmonic.

T=1355 N/m k=0.4 σ=3.125						
Mode mn	Homogeneous membrane fcalc (Hz)	Damped Homogeneous membrane fcalc (Hz)	Bi-density membrane fcalc (Hz)	Damped Bi-density membrane fcalc (Hz)	Damped Homogeneous membrane fmn/f11	Damped Bi-density membrane fmn/f11
01	133.7	53	35.1	33.4	0.55	0.51
11	179.2	97	68	65.8	1	1
21	222.6	139	103.5	99.9	1.44	1.52
02	244.8	151	107.2	104.5	1.56	1.59
31	264.8	180	139.3	135.5	1.86	2.06
12	293.7	201	143.9	141.3	2.07	2.15
41	306.1	221	174.4	7.2*	2.28	-
22	340.6	248	180.7	171.6	2.56	2.61
03	355.0	256	169.3	165.5	2.64	2.52

** Anomalous case currently being looked into*

Figure 2.3: Comparison of the eigenfrequencies for the damped and undamped cases of homogeneous and bi-density membranes. It is seen that harmonic ratios are obtained for the damped bi-density membrane.



(a) Mode 01 (b) Mode 11 (c) Mode 21

(d) Mode 02 (e) Mode 31 (f) Mode 12

(g) Mode 41 (h) Mode 22

Figure 2.4: Eigenmodes for the damped bi-density case

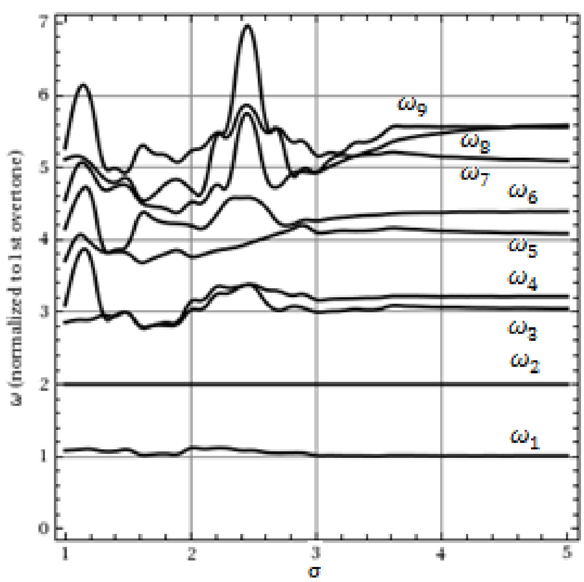


Figure 2.5: Variation of the eigenfrequencies with mass density ratio σ for the concentric case. Frequencies are normalized by the first overtone. Notice that for the middle region the harmonicity is lifted due to radiation effects.

CHAPTER 3

A NUMERICAL APPROACH

3.1 Membrane with smoothly varying density

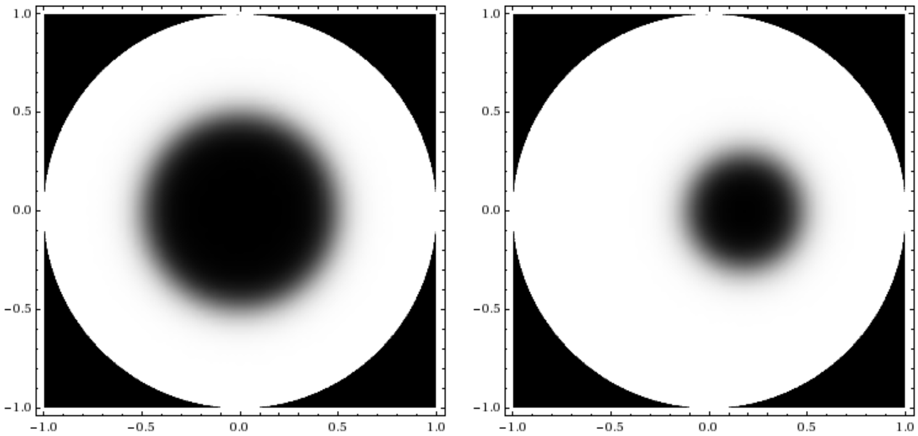
The Tabla drumhead is approximated as a circular membrane of unit radius, with a non uniform areal density $\rho(\mathbf{r})$. The concentric and the eccentric cases are modelled as,

$$\rho(r,\theta) = 1 + \frac{(\sigma^2 - 1)}{2}\left[1 - tanh\left(\frac{R(r,\theta) - k}{\xi}\right)\right]$$

where

$$R(r,\theta) = \sqrt{(r\cos\theta - \epsilon)^2 + (r\sin\theta)^2}$$

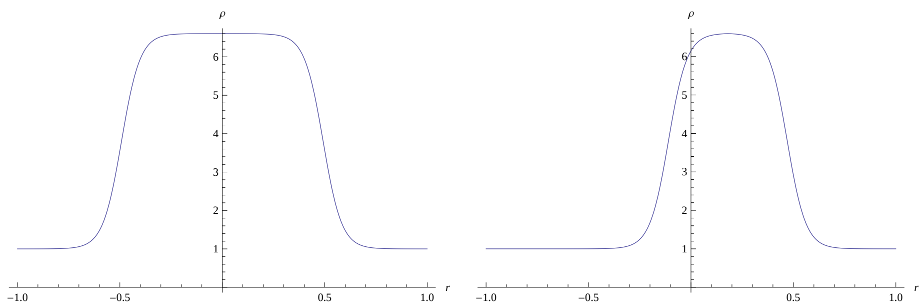
The function changes smoothly from unit density at $r = 1$ to σ^2 at the center of the region. The change occurs over a region of width ξ along the circle whose equation in polar coordinates is $r = R(r,\theta)$. Concentric loading in the case of *dayan* is obtained by setting $\epsilon = 0$ which gives a radially symmetric loading. For $\epsilon > 0$ the one obtains loading that is displaced from the center by a distance ϵ corresponding to the eccentric *bayan*. The parameter k^2 represents the ratio of the areas of loaded and unloaded regions for $\xi \ll 1$. The uniform membrane is retrieved by setting $\sigma = 1$ and considering the case $\xi \to 0$ with other parameters fixed the bi-density model is recovered. The model avoids the abrupt change in density associated with the bi-density model and allows gradual decrease in the density of the loaded region.



(a) Variation of the areal density of the membrane for $\epsilon = 0$. The density plot is for $\sigma = 2.57$, $k = 0.492$ and $\xi = 0.091$. This form is used to model the *dayan*

(b) Variation of the areal density of the membrane for $\epsilon > 0$. The density plot is for $\epsilon = 0.18$, $\sigma = 2.57$, $k = 0.29$ and $\xi = 0.091$. This form is used to model the *bayan*

Figure 3.1: The smoothly varying density model for *dayan* and *bayan*



(a) (b)

Figure 3.2: The variation of areal density with the radial coordinate r for *dayan* and *bayan* respectively

3.2 Lossy wave equation for non-uniform membrane

Loss modelling in membranes is a very involved matter- there exist various sources of loss like radiation, thermoelasticity and viscoelasticity; see [10] for a detailed picture of such effects. Real membranes display a rather complex loss characteristic-the various frequency components do not decay at the same rate. Loss generally increases with frequency, leading to a sound with wide band attack, decaying to a few harmonics. One model for the frequency dependent loss is given by the following PDE [11]

$$u_{tt} = \gamma^2 \nabla^2 u - 2\sigma_0 u_t + 2\sigma_1 \nabla^2 u_t$$

$$u(r = 1, \theta, t) = 0 \quad \text{; clamped boundary condition}$$

where $\gamma = \gamma(\mathbf{r})$ is the scaled density with σ_0 and σ_1 as the damping parameters. The term with coefficient $\sigma_0 \geq 0$ on its own gives rise to bulk frequency-independent damping. The mixed derivative term with $\sigma_1 \geq 0$ models the frequency dependent loss.

Assuming a solution of the form $u(r, \theta, t) = \psi_{mn}(r, \theta)\, e^{\imath\omega t}$ the equation is transformed to

$$-\omega_{mn}^2 \Psi_{mn} = \gamma^2\, \nabla^2\, \Psi_{mn} - 2\,\imath\, \omega_{mn}(\, \sigma_0 - 2\, \sigma_1\, \nabla^2)\, \Psi_{mn} \quad \text{or}$$

$$\left(\omega_{mn}^2 - 2\,\imath\, \omega_{mn}(\, \sigma_0 - 2\, \sigma_1\, \nabla^2) + \gamma^2\, \nabla^2\right) \Psi_{mn} = 0$$

Here Ψ_{mn} is an eigenmode with m nodal lines and n nodal contours. As the density varies with position a generalized eigenvalue problem is expected. However due to terms with first order time derivatives a *Quadratic Generalized Eigenvalue* problem is obtained. In the next section a high resolution spectral collocation method based on Fourier-Chebyshev expansions is presented for numerical evaluation of eigenvalues and eigenmodes of the lossy smoothly non-uniform membrane.

3.3 Computing the Eigenspectra

The *Quadratic Generalized Eigenvalue* problem obtained for the lossy smoothly non-uniform membrane can be solved using methods of varying degrees of accuracy and sophistication. Applicable techniques include Finite difference (FD) and the Finite-element (FE) and more recently **Spectral Methods**. Finite-element methods are particularly suitable for problems with very complex geometries while Finite difference methods perform well for a broad class of problems with moderately complex domains. Spectral methods are usually the best option when the domain is simple and the data defining the problem is smooth; both conditions are met in the formulation of the PDE for the lossy membrane. They are also highly accurate and offer spectral convergence with the least computational expense. A Fourier-Chebyshev spectral collocation technique, therefore, is used to study the *Quadratic Generalized Eigenvalue* problem and is described below.

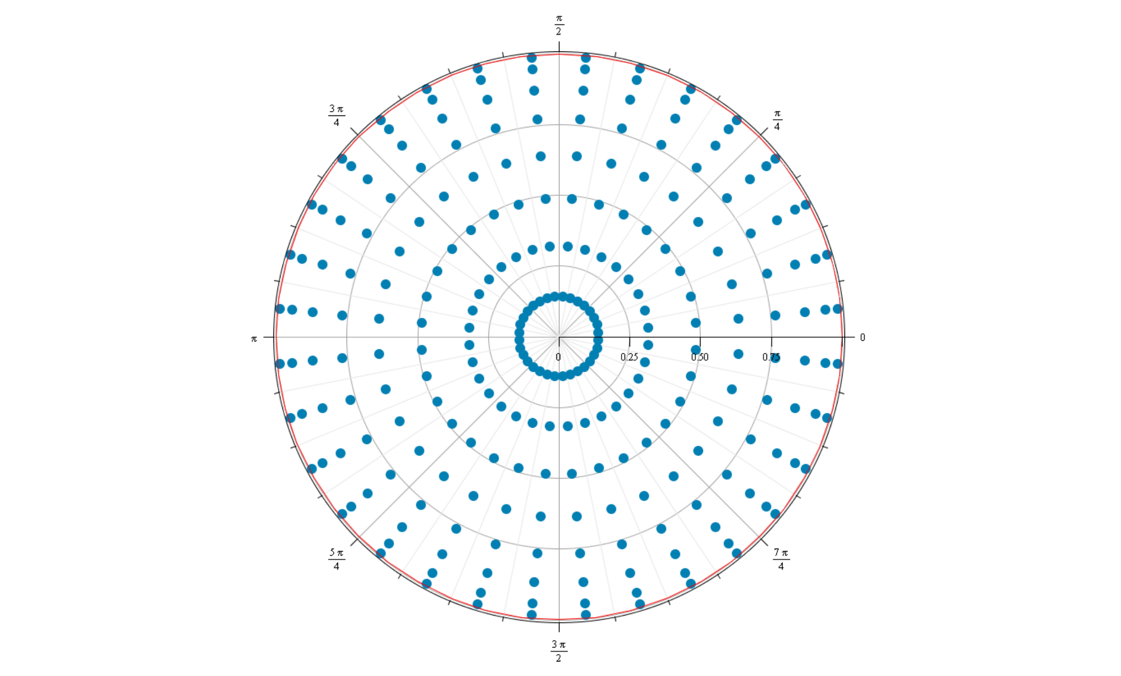


Figure 3.3: The Fourier-Chebyshev Grid for polar coordinates. Note the clustering of points at the edges.

The method proceeds by approximating the solution as a linear combination of very smooth basis functions that are orthogonal. The expansions are based on global functions and the solution is in the form of a global interpolant. The expansion coefficients are determined by *collocation* (other methods include *Galerkin* and *Tau*) wherein the the residual is made zero at suitably chosen collocation points or nodes. The interpolant can be differentiated exactly with the derivatives being represented by spectral differentiation matrices. Multiplying a spectral differentiation matrix with a vector of function values returns the derivative. For multidimensional problems on a tensor product spectral grid *Kronecker products* of operators are used; for example, partial derivatives are obtained by Kronecker products of differentiation matrices corresponding to each independent variable.

In Fourier-Chebyshev spectral collocation, a Fourier expansion is used for the angular coordinate $\theta \in [0\ 2\pi]$ and Chebyshev expansion is used for the radial coordinate $r \in [0\ 1]$. A method proposed by Fornberg is used to obtain Chebyshev expansion for functions on $[0\ 1]$ instead of the usual $[-1\ 1]$ using the implementation by Trefethen [12]. The Laplacian in polar coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}$$

is represented by the Fourier-Chebyshev differentiation matrix

$$\mathbf{L} = (\mathbf{D_1} + \mathbf{RE_1}) \otimes \mathbf{I_l} + (\mathbf{D_2} + \mathbf{RE_2}) \otimes \mathbf{I_r} + \mathbf{R^2} \otimes \mathbf{D_\theta^2}$$

The matrices above are representative of partial derivatives on the grid for N_r(odd) and N_θ(even) Chebyshev and Fourier collocation points respectively . Matrices $\mathbf{D_1}$ and $\mathbf{D_2}$ together represent ∂_r^2. Terms with $\mathbf{E_1}$ and $\mathbf{E_2}$ represent $r^{-1}\partial_r$ and the last term with $\mathbf{D}_\theta$ represents $r^{-2}\partial_\theta^2$. $\mathbf{R}$ is the diagonal matrix $diag(r_j^{-1})$, $1 \leq j \leq \frac{N_r-1}{2}$. The two identity matrices,

$$\mathbf{I_l} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \quad \mathbf{I_r} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}$$

are formed out of the $N_{\theta/2} \times N_{\theta/2}$ identity matrix I. For further details see [12]. With the loading function $\rho(r,\theta)$ represented by a matrix $\mathbf{B}$ the *Quadratic Generalized Eigenvalue* problem can be formulated as a matrix equation,

$$\left(\lambda^2 \mathbf{A_2} + \lambda \mathbf{A_1} + \mathbf{A_0}\right) \mathbf{\Psi} = 0$$

where $\mathbf{A_0} = \mathbf{L}$, $\mathbf{A_1} = -2\,\imath\,(\,\sigma_0 \mathbf{B}^2 - \sigma_1 \mathbf{LB})$ and $\mathbf{A_2} = \mathbf{B}$

The eigenvalues λ_{mn} and eigenvectors Ψ_{mn} are obtained using the MATLAB function polyeig which is based on QZ factorization algorithm.

3.4 Results

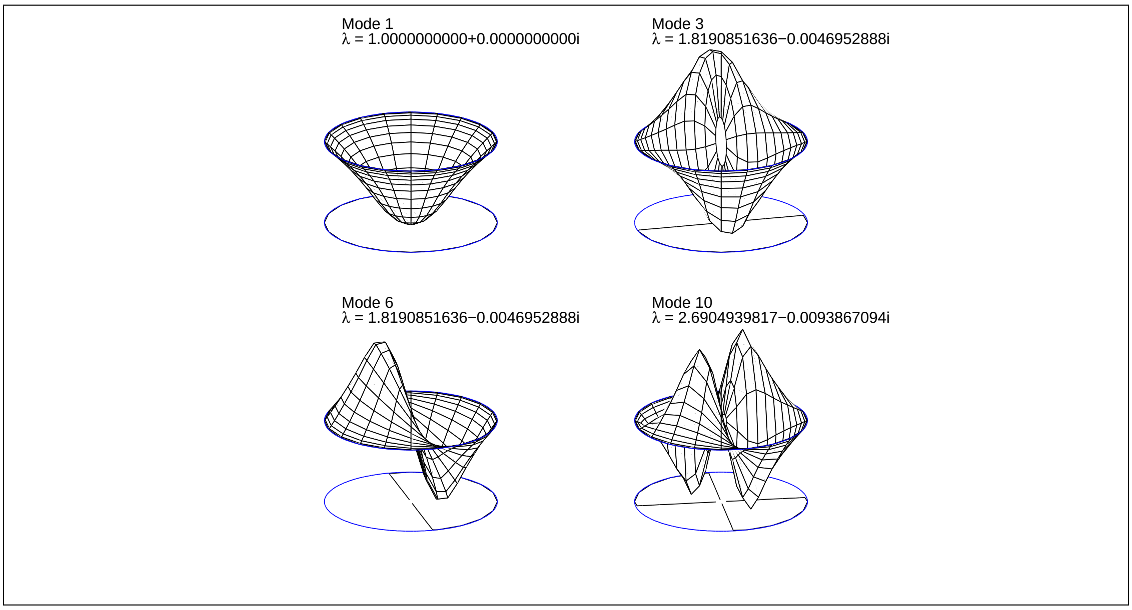


Figure 3.4: Damped Eigenmodes for the concentric case

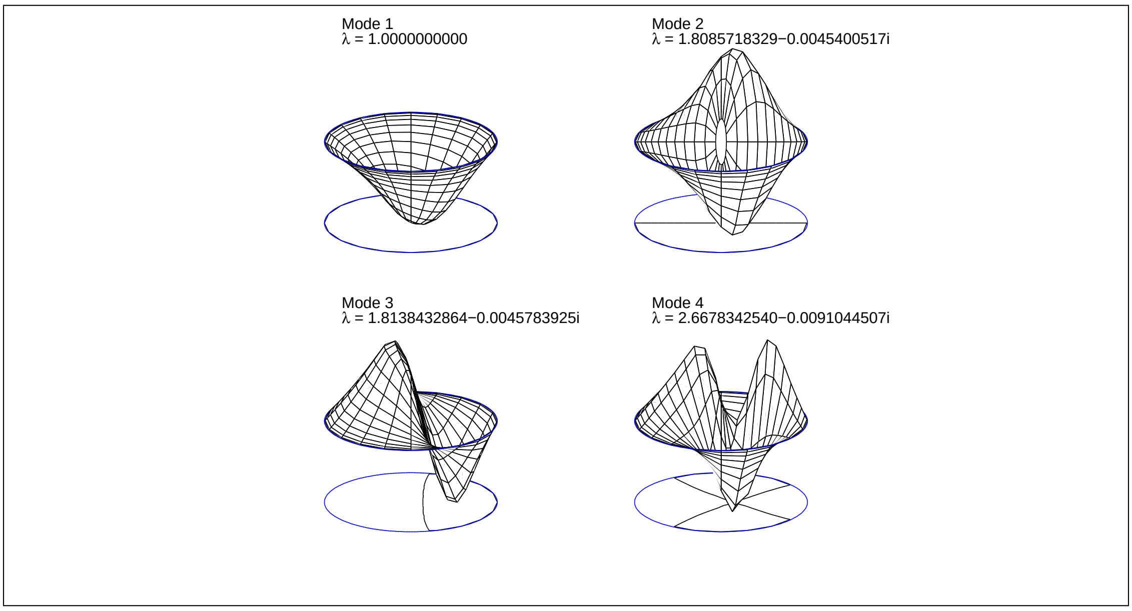


Figure 3.5: Damped Eigenmodes for the eccentric case. Observe the generelization of nodal lines to nodal contours.

CHAPTER 4

NUMERICAL SOUND SYNTHESIS

Physical modelling synthesis, which has developed recently, involves a physical description of musical instrument as the starting point. With a physical model for the Tabla membrane now in place an attempt is made at numerical sound synthesis. The idea is to solve the set of equations through a numerical approximation and yield an output waveform subject to some input excitation.

4.1 Modal Synthesis

In the field of numerical sound synthesis the *Modal Synthesis* approach is based on frequency domain description of the vibration of distributed objects. The complex dynamic behavior of a vibrating object is decomposed into contributions from a set of modes whose spatial parts are eigenfunctions (dependent on boundary conditions) of the given problem at hand. Each such mode oscillates at a single complex frequency (complex eigenvalues occurring as conjugate pair). Modal synthesis forms the basis of many commercial sound synthesis software packages.

The physical model may be described in terms of two sets of data

1. the PDE system and associated boundary conditions, with information regarding material properties and geometry
2. excitation information, including initial conditions and readout locations

The first set of information is used to determine the modal shapes and frequencies of vibration; this involves essentially the solution of an eigenvalue problem. With the determination of the complex eigenfunctions and eigenfrequencies for the lossy membrane this step in sound synthesis is already accomplished. The second set of informations is then employed - the initial condition or excitation is expanded onto the set of modal functions (from which an orthogonal set can be derived) through an inner product, giving a set of weighting coefficients. The weighted combination of the modal functions evolves with each mode evolving at its own frequency. To obtain a sound output, in the simplest case, the modal functions are projected onto a delta function at a given location on the object.

4.2 Mathematical formulation

The initial displacement of the membrane (in Dirac's *bra-ket* notation) $|\mathbf{\Psi_{in}}\rangle$ can be expanded in terms of the set of normalized complex eigenfunctions $|\mathbf{\Psi_i}\rangle$ $(i = 1, \ldots n)$ obtained for the lossy membrane. In this formulation $|\mathbf{\Psi_i}\rangle$ represents a vector of complex values. Though the eigenfunctions are *non-orthogonal* they are linearly independent so an orthonormal basis can be derived. The decomposition in terms of non-orthogonal complex eigenfunctions is

$$|\mathbf{\Psi_{in}}\rangle = a_1|\mathbf{\Psi_1}\rangle + a_2|\mathbf{\Psi_2}\rangle + \ldots + a_n|\mathbf{\Psi_n}\rangle$$

where a_i's are complex weighting coefficients and $|\mathbf{\Psi_i}\rangle$ are the eigenfunctions. Taking inner product of the above equation with each $|\mathbf{\Psi_i}\rangle$ gives us a set of linear equations for the weighting coefficients

$$\begin{aligned}
\langle\mathbf{\Psi_1}|\mathbf{\Psi_{in}}\rangle &= a_1\langle\mathbf{\Psi_1}|\mathbf{\Psi_1}\rangle + a_2\langle\mathbf{\Psi_1}|\mathbf{\Psi_2}\rangle + \ldots + a_n\langle\mathbf{\Psi_1}|\mathbf{\Psi_n}\rangle \\
\langle\mathbf{\Psi_2}|\mathbf{\Psi_{in}}\rangle &= a_1\langle\mathbf{\Psi_2}|\mathbf{\Psi_1}\rangle + a_2\langle\mathbf{\Psi_2}|\mathbf{\Psi_2}\rangle + \ldots + a_n\langle\mathbf{\Psi_2}|\mathbf{\Psi_n}\rangle \\
&\vdots \\
\langle\mathbf{\Psi_n}|\mathbf{\Psi_{in}}\rangle &= a_1\langle\mathbf{\Psi_n}|\mathbf{\Psi_1}\rangle + a_2\langle\mathbf{\Psi_n}|\mathbf{\Psi_2}\rangle + \ldots + a_n\langle\mathbf{\Psi_n}|\mathbf{\Psi_n}\rangle
\end{aligned}$$

The weighting coefficients are determined as solutions of a system of n linear equations

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle\mathbf{\Psi_1}|\mathbf{\Psi_1}\rangle & \cdots & \langle\mathbf{\Psi_1}|\mathbf{\Psi_n}\rangle \\ \vdots & \ddots & \vdots \\ \langle\mathbf{\Psi_n}|\mathbf{\Psi_1}\rangle & \cdots & \langle\mathbf{\Psi_n}|\mathbf{\Psi_n}\rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle\mathbf{\Psi_1}|\mathbf{\Psi_{in}}\rangle \\ \vdots \\ \langle\mathbf{\Psi_n}|\mathbf{\Psi_{in}}\rangle \end{bmatrix}$$

by determining the coefficients a_i $(i = 1, \ldots n)$ the projection of the initial membrane shape onto the modal shapes for the membrane is obtained. This corresponds to the *pluck* condition where the initial shape is specified with no initial velocity.

4.2.1 Raised cosine initial condition

The wave equation is a second order in time and like all such PDEs must be initialized with two conditions. Normally the initial shape and velocity profile are specified i.e we set

$$\begin{aligned} u(r, \phi, 0) &= u_0(r, \phi) \qquad \textit{pluck condition} \\ (\partial u / \partial t)\,(r, \phi) &= v_0(r, \phi) \qquad \textit{strike condition} \end{aligned}$$

A useful distribution in such cases is the 2D raised cosine, of the form

$$c_{rc}(r, \phi) = \begin{cases} \frac{c_0}{2}\left(1 + \cos\left(\pi\sqrt{r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0)}/r_{hw}\right)\right), & |\mathbf{r} - \mathbf{r_0}| \le r_{hw} \\ 0 & |\mathbf{r} - \mathbf{r_0}| > r_{hw} \end{cases}$$

which has amplitude c_0, half width r_{hw} and is centered at $\mathbf{r_0} = (r_0, \phi_0)$. Such a distribution can be used to model both plucks and strikes. For our purposes we assume a raised cosine distribution as the initial displacement of the membrane $|\mathbf{\Psi_{in}}\rangle$.

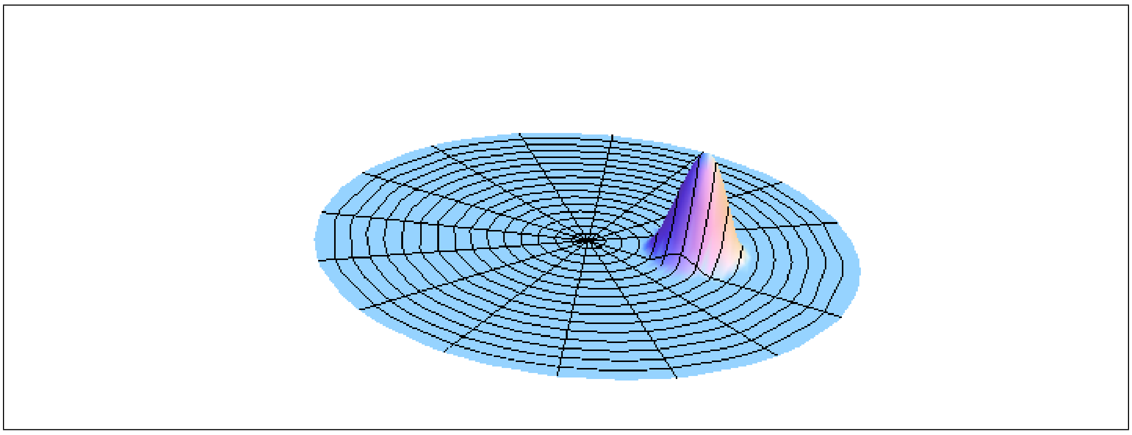


Figure 4.1: The raised cosine initial condition.

4.2.2 Propagation in time

With $|\mathbf{\Psi_{in}}\rangle$ set as 2D raised cosine and the corresponding weighting coefficients a_i's for modal projection one can write

$$|\mathbf{\Psi(t)}\rangle = a_1|\mathbf{\Psi_1}\rangle e^{\imath\omega_1 t} + a_2|\mathbf{\Psi_2}\rangle e^{\imath\omega_2 t} + \ldots + a_n|\mathbf{\Psi_n}\rangle e^{\imath\omega_n t}$$

$$|\mathbf{\Psi(t)}\rangle = \begin{bmatrix} e^{\imath\omega_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\imath\omega_n t} \end{bmatrix} \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} |\mathbf{\Psi_1}\rangle \\ \vdots \\ |\mathbf{\Psi_n}\rangle \end{bmatrix}$$

with matrix $\mathbf{\Omega}$ and $\mathbf{A}$ given by

$$\mathbf{\Omega} = \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_n \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix}$$

$$|\mathbf{\Psi(t)}\rangle = e^{\mathbf{\Omega}\,t}\ \mathbf{A} \begin{bmatrix} |\mathbf{\Psi_1}\rangle \\ \vdots \\ |\mathbf{\Psi_n}\rangle \end{bmatrix}$$

The matrix formulation of the 2D wave propagation on a lossy smoothly varying non-uniform membrane is now complete. We can now simulate a plucked or struck membrane and see how it evolves in time. The sound in the form of output waveform can be readout from any suitable point on the membrane.

4.3 Results and Discussion

As a beginning in sound synthesis we first analyze using MATLAB the characteristic *bol 'Tun'* played on the *dayan*. The output waveform and the Power Spectral Density PSD for the actual recording are plotted. A noticeable peak at the dominant frequency is observed. This peak frequency corresponds to the eigenfrequency of the lowest excited mode.

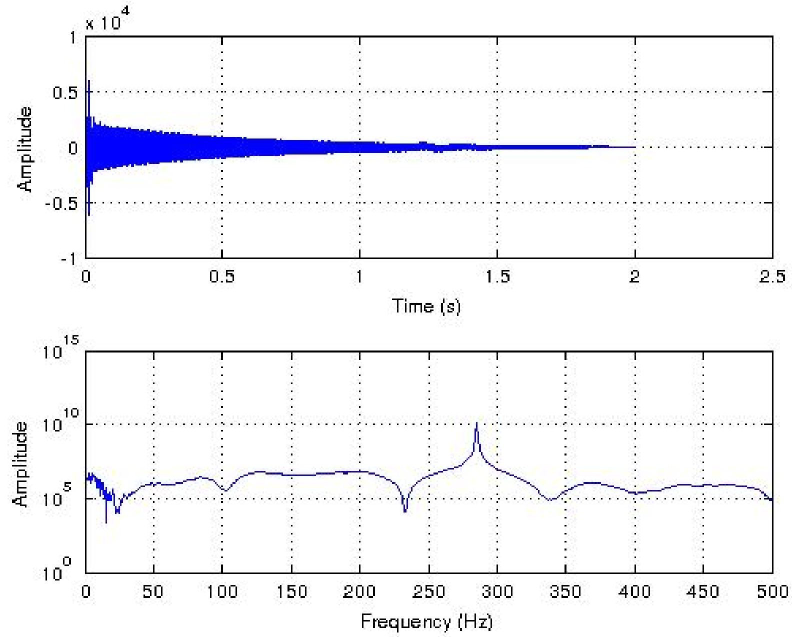


Figure 4.2: The raw output waveform and Power Spectral Density for a recording of Tabla bol *'Tun'*. Peak corresponding to the fundamental frequency is clearly visible.

To investigate the the decay profile for the the various frequencies a spectrogram is employed. Fig 4.3 shows the spectrogram of a recorded sound sample with frequency and time on the y and x axis respectively. The intensity which varies over time is depicted through variation in color for a particular frequency.

The damping parameters σ_0 and σ_1 are estimated using the decay profiles of eigenfrequencies (corresponding to peaks in PSD plot). Using these damping parameters for Modal Synthesis one obtains numerically synthesized sound for our model of the Tabla membrane. Fig. 4.4 shows an input excitation in the form of a raised cosine distribution propagating in time. A fixed readout location on the membrane is chosen to read the output waveform shown in Fig 4.5.

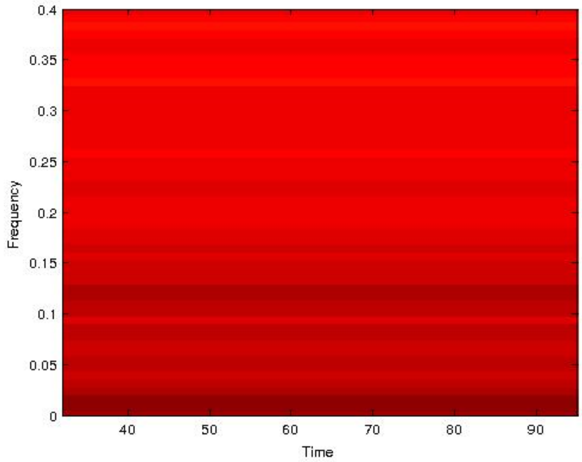


Figure 4.3: The spectrogram is a visual representation of the decay profile of sinusoids with corresponding frequencies. As the sinusoid in the Fourier decomposition of the signal decays the line corresponding to its frequency gradually lightens.

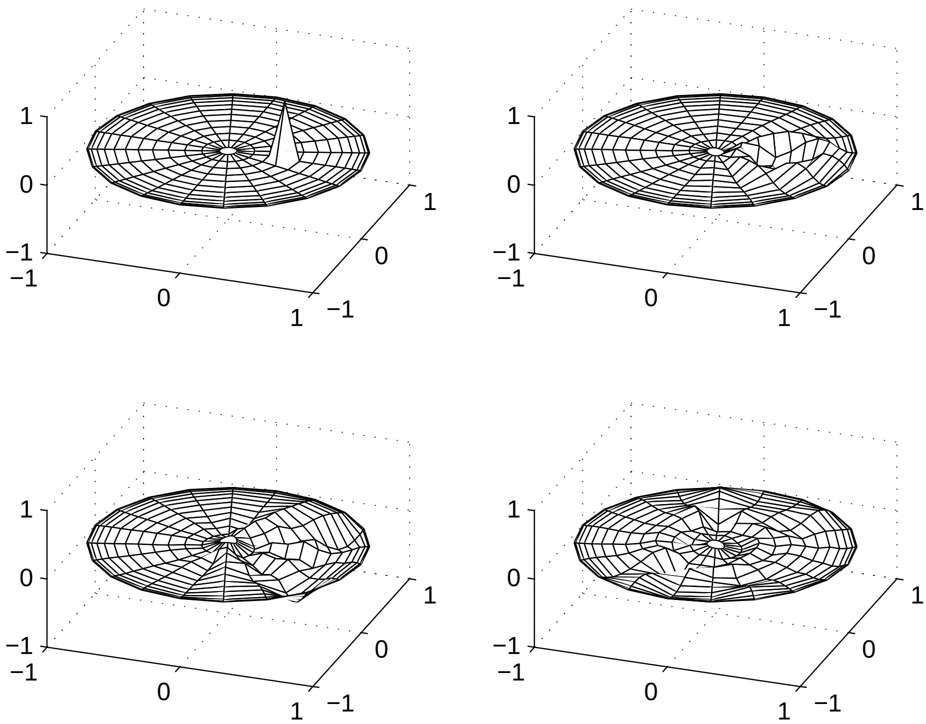


Figure 4.4: The raised cosine distribution corresponding to the *pluck* condition propagating in time over the membrane. The coarseness is due to the grid; the interpolant being spectrally accurate.

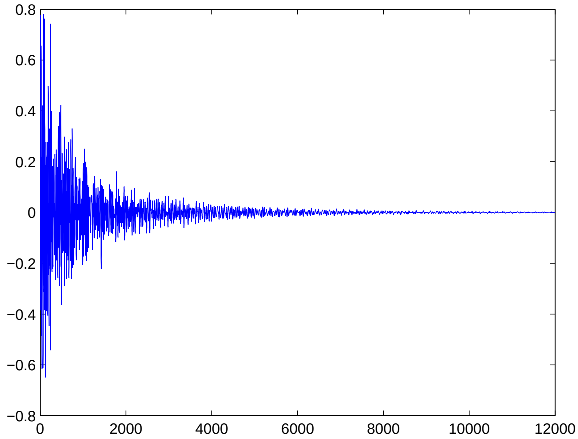


Figure 4.5: Raw waveform output of Modal Synthesis at a particular readout location on the membrane. The decay profile is seen to be similar to that derived from actual recording of the Tabla bol *Tun*

CHAPTER 5

SUMMARY AND CONCLUSION

Through the work carried out for this thesis a physical model for the *Tabla* has been realized. The radiation damping effects for the *Tabla* drumhead modeled as a bi-density membrane were characterized through an analytic formulation. It was found that air-loading results in significant lowering of the eigenfrequencies for a homogeneous membrane. However for a bi-density membrane the stepped variation in density is the major factor in lowering frequencies and air-loading results in only marginal reduction.

Next a smoothly non-uniform membrane model was considered which modeled both concentric and eccentric loading. Phenomenological damping terms were added to the PDE formulation to account for radiation and frequency dependent loss. The eigenfrequencies and eigenfunctions were determined through a high resolution Fourier-Chebyshev spectral collocation method. The damping parameters were approximately determined from analysis of actual sound sample of the *'Tun'* .

Modal synthesis was then used to generate the transient response of the system subject to an input excitation. The response was read at a fixed location on the membrane to generate the characteristic sound of *'Tun'* a *bol* played on *dayan*.

In conclusion a high fidelity rendition of a particular *Tabla bol* was obtained numerically leaving scope for numerical synthesis of the entire spectrum of such sounds from the instrument.

CHAPTER 6

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